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Localized excitations of the (2 + 1)-dimensional sine-Gordon system

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Abstract

The (2 + 1)-dimensional sine-Gordon system is investigated via a generalized bilinear operator representation. Plateau, basin, bowl and saddle-type ring solitons are thereby constructed. It is indicated how iterated Moutard transformations may be employed to extend the range of the method.

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1. Introduction

Ring-like solitonic phenomena have been widely investigated. However, an impediment to the analytic study of their interaction properties in sine-Gordon models in three dimensions has been the non-integrability of $\left(\partial_t^2 - \nabla^2\right)\psi = \sin\psi$. The Davey–Stewartson and Nizhnik–Veselov–Novikov systems are well established as integrable symmetric (2 + 1)-dimensional extensions of the nonlinear Schrödinger and Korteweg–de Vries equations. The corresponding integrable extension to three dimensions of the classical sine-Gordon equation is represented by the (2 + 1)-dimensional sine-Gordon (2DsG) system as introduced by Konopelchenko and Rogers [1, 2].

Localized solutions of (2 + 1)-dimensional soliton equations have proved of considerable interest. Indeed, the discovery via Bäcklund transformations by Boiti *et al* [3] of 'dromion' type coherent solutions of the Davey–Stewartson I system provided renewed interest in the study of three-dimensional soliton systems. Here, our concern is with localized excitations of the 2DsG system.

Recently in [4, 5], a multi-linear form approach was introduced whereby new localized solutions may be constructed for a range of nonlinear systems. This includes, *inter alia*, not only Davey–Stewartson and Nizhnik–Veselov–Novikov equations but also Broer–Kaup–Kuperschmidt and the generalized (N + M)-component AKNS systems. Here, that procedure is adapted to the 2DsG system. A diversity of ring-like soliton solutions that exhibit complete elastic interaction is thereby generated.

2. The (2 + 1)-dimensional sine-Gordon system

In 1991, Konopelchenko and Rogers [1] constructed a (2 + 1)-dimensional master soliton system via a reinterpretation and generalization of a class of infinitesimal Bäcklund transformations originally introduced in a gasdynamics context by Loewner [6]. This (LKR) system is characterized by its admittance of a matrix parametrization. A particular reduction leads to a symmetric integrable extension of the classical sine-Gordon equation, namely,

$$\begin{aligned}&\left(\frac{\phi_x}{\sin\theta}\right)_x - \left(\frac{\phi_y}{\sin\theta}\right)_y + \frac{\phi_y\theta_x - \phi_x\theta_y}{\sin^2\theta} = 0\\ &\left(\frac{\phi'_x}{\sin\theta}\right)_x - \left(\frac{\phi'_y}{\sin\theta}\right)_y + \frac{\phi'_y\theta_x - \phi'_x\theta_y}{\sin^2\theta} = 0\\ &\theta_t = \phi + \phi'.\end{aligned} \tag{2.1}$$

The classical (1 + 1)-dimensional sG equation in lines of curvature coordinates is retrieved in the reductions with $\phi' = \theta_y = \phi_y = 0$. The 2DsG system (2.1) is generated as the compatibility condition of the particular LKR triad [2]

$$\begin{aligned}&\left[I\partial_x + \begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix}\partial_y\right]\psi = 0\\ &\left[I\partial_t\partial_y + \frac{1}{2}\begin{pmatrix}0 & \theta_t\\ -\theta_t & 0\end{pmatrix}\partial_y - \frac{1}{2\sin\theta}\begin{pmatrix}\phi_y\cos\theta - \phi_x & -\phi'_y\sin\theta\\ \phi_y\sin\theta & \phi'_y + \phi'_x\cos\theta\end{pmatrix}\right]\psi = 0\\ &\left[I\partial_t\partial_x + \frac{1}{2}\begin{pmatrix}0 & \theta_t\\ -\theta_t & 0\end{pmatrix}\partial_x - \frac{1}{2\sin\theta}\begin{pmatrix}\phi_x\cos\theta - \phi_y & -\phi'_x\sin\theta\\ \phi_x\sin\theta & \phi'_x + \phi'_y\cos\theta\end{pmatrix}\right]\psi = 0.\end{aligned} \tag{2.2}$$

Alternative representations of the 2DsG system prove convenient. Thus, on introduction of the new independent variables

$$\xi = \tfrac{1}{2}(y - x) \qquad \eta = \tfrac{1}{2}(y + x)$$

the system may be rewritten as

$$\theta_{\xi\eta t} + \tfrac{1}{2}\theta_\eta\rho_\xi + \tfrac{1}{2}\theta_\xi\rho_\eta = 0 \qquad \rho_{\xi\eta} = \tfrac{1}{2}(\theta_\xi\theta_\eta)_t \tag{2.3}$$

where

$$\rho_\xi = \frac{\phi'_\eta - \phi_\eta - \theta_{\eta t}\cos\theta}{\sin\theta} \qquad \rho_\eta = \frac{\phi_\xi - \phi'_\xi - \theta_{\xi t}\cos\theta}{\sin\theta}. \tag{2.4}$$

The introduction of new dependent variables u, v into (2.4) via the relation

$$\rho = 2v_t \qquad \theta = 2u \tag{2.5}$$

leads to a compact version of the 2DsG system, namely

$$u_{\xi\eta t} + u_\eta v_{\xi t} + u_\xi v_{\eta t} = 0 \qquad v_{\xi\eta} = u_\xi u_\eta + v_0 \tag{2.6}$$

where the arbitrary function $v_0 \equiv v_0(\xi, \eta)$ is here set to zero as in [7, 8]. The corresponding Lax pair for the representation (2.6) with $v_0 = 0$ reads

$$\begin{pmatrix}\partial_\xi & u_\xi\\ -u_\eta & \partial_\eta\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = 0 \qquad \begin{pmatrix}\partial_\eta\partial_t + v_{\eta t} & u_\eta\partial_t\\ -u_\xi\partial_t & \partial_\xi\partial_t + v_{\xi t}\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = 0. \tag{2.7}$$

Since the discovery of the 2DsG system a decade ago, it has been the subject of wide investigation not least because of its rich symmetry structure. A Bäcklund transformation was constructed in [9] and certain coherent solitonic solutions thereby derived. Solitonic solutions of the important reduction

$$\theta_{xyt} - \theta_x\theta_{yt}\cot\theta + \theta_y\theta_{xt}\tan\theta = 0 \tag{2.8}$$

that arises in connection with triple orthogonal systems of surfaces, have been investigated by Nimmo [10, 11]. Doubly periodic wave solutions have been constructed by Chow [16]. Localized solutions of the 2DsG system were constructed via a binary Darboux transformation by Schief [12]. In [8], Nimmo and Schief constructed nonlinear superposition principles and an associated integrable discretization of the 2DsG system. Localized solutions of the model with nontrivial boundaries have been constructed by Dubrovsky and Konopelchenko [13] and Dubrovsky and Formusatik [15]. Geometric aspects of the 2DsG system were investigated by Schief [17]. Extensive symmetry group analysis of the system has been conducted by both Clarkson *et al* [7] and Lou [18]. Radha and Lakshmanan [19] studied the Painlevé property for the 2DsG system and have constructed dromion solutions. It emerges that the system contains particular reductions to the PI, PIII and PV transcendents. In physical terms, it also contains the important pumped Maxwell–Bloch system.

The 2DsG system has been discussed in the general context of three-dimensional integrable systems in the monograph by Konopelchenko [20]. However, its complex structure is such that a complete understanding of the model is yet to be achieved. In particular, the question as to its admittance of saddle-type ring solitons with completely elastic interaction properties has remained open. Here, the Hirota-operator based approach adopted in [5] is applied to the 2DsG system to construct new solutions both with and without completely elastic interaction properties. Plateau, basin, bowl and saddle-type ring solitons are thereby constructed which exhibit completely elastic interaction properties. It is demonstrated how the range of the procedure may be extended by the application of iterated Moutard transformations.

3. A generalized bilinear representation for the 2DsG system

The starting point for the procedure is to determine an appropriate multi-linear representation for the system under consideration. In the present case of the 2DsG system (2.6) with $v_0 = 0$, if we set

$$u = \pm\mathrm{i}\ln\frac{f}{g} + u_1 \qquad v = \ln(fg) + v_1 \tag{3.1}$$

where $\{u_1, v_1\}$ is an arbitrary seed solution, then we obtain the multi-linear representation

$$\begin{aligned}
&\pm\mathrm{i} fg[D_\xi D_\eta D_t + v_{1\xi t} D_\eta + v_{1\eta t} D_\xi] f\cdot g \\
&\qquad + u_{1\eta}\left[(D_t f\cdot g)(D_\xi f\cdot g) + \tfrac{1}{2} f^2 D_\xi D_t g\cdot g + \tfrac{1}{2} g^2 D_\xi D_t f\cdot f\right] \\
&\qquad + u_{1\xi}\left[(D_t f\cdot g)(D_\eta f\cdot g) + \tfrac{1}{2} f^2 D_\eta D_t g\cdot g + \tfrac{1}{2} g^2 D_\eta D_t f\cdot f\right] = 0 \\
&[\pm\mathrm{i} D_\xi D_\eta + u_{1\eta} D_\xi + u_{1\xi} D_\eta] f\cdot g = 0
\end{aligned} \tag{3.2}$$

wherein the Hirota's bilinear operators D_ξ, D_η, D_t are defined by

$$D_\xi^n D_\eta^m D_t^k f\cdot g \equiv \partial_{\epsilon_1}^n \partial_{\epsilon_2}^m \partial_{\epsilon_3}^k f(\xi+\epsilon_1, \eta+\epsilon_2, t+\epsilon_3) g(\xi-\epsilon_1, \eta-\epsilon_2, t-\epsilon_3)\big|_{\epsilon_1=0,\epsilon_2=0,\epsilon_3=0}. \tag{3.3}$$

If the seed solution $\{u_1, v_1\}$ is set to zero (or constant), a known bilinear representation of the 2DsG system is retrieved [11].

Here to exploit the representation (3.2), we select a seed solution of the system (2.6) with $v_0 = 0$ in the form

$$u_1 = 0 \qquad v_1 = V_1(\xi, t) + V_2(\eta, t) \tag{3.4}$$

with $V_1(\xi, t) \equiv V_1$ and $V_2(\eta, t) \equiv V_2$ being arbitrary functions. The multi-linear system (3.2) then degenerates to a bilinear system in the form

$$[D_\xi D_\eta D_t + V_{1\xi t} D_\eta + V_{2\eta t} D_\xi] f\cdot g = 0 \qquad D_\xi D_\eta f\cdot g = 0. \tag{3.5}$$

To obtain solutions of this bilinear system, we adopt the ansatz

$$f = a_0 + a_1 p + a_2 q + a_3 pq \qquad g = b_0 + b_1 p + b_2 q + b_3 pq \tag{3.6}$$

where a_0 and b_0 are arbitrary constants, $p \equiv p(\xi, t), q \equiv q(\eta, t), a_1 \equiv a_1(t), a_2 \equiv a_2(t)$, $a_3 \equiv a_3(t), b_1 \equiv b_1(t), b_2 \equiv b_2(t)$ and $b_3 \equiv b_3(t)$ are functions of their indicated arguments. Detailed calculations via computer algebra (Maple) establish that the ansatz (3.6) allows solution of the bilinear system (3.5) if the functions V_1 and V_2 are determined by

$$V_{1\xi t} = \frac{(2A_3 p_t + (A_1 - c_1)a_0)p_\xi - 2(A_3 p + a_0 A_2)p_{\xi t}}{a_0(2A_2 p + A_4) + A_3 p^2} \tag{3.7}$$

$$V_{2\eta t} = \frac{(a_0(c_1 + A_1) - 2A_5 q_t)q_\eta + 2(A_5 q - a_0 A_7)q_{\eta t}}{a_0(A_6 + 2A_7 q) - A_5 q^2} \tag{3.8}$$

where $c_1 \equiv c_1(t)$ is an arbitrary function of t,

$$\begin{aligned} &A_1 = b_2 a_{1t} + b_1 a_{2t} - b_0 a_{3t} \qquad A_2 = a_3 b_0 - b_2 a_1 \\ &A_3 = a_3 b_1 a_0 - a_1^2 b_2 - a_1 a_2 b_1 + a_3 a_1 b_0 \qquad A_4 = b_0 a_2 - a_0 b_2 \\ &A_5 = -a_3 a_2 b_0 + a_2^2 b_1 + a_2 b_2 a_1 - a_0 b_2 a_3 \qquad A_6 = b_0 a_1 - a_0 b_1 \\ &A_7 = b_0 a_3 - b_1 a_2 \end{aligned} \tag{3.9}$$

and the function b_3 is fixed via

$$b_3 = \frac{1}{a_0}(b_2 a_1 + b_1 a_2 - b_0 a_3). \tag{3.10}$$

The quantities $2\mathrm{i}u_{\xi\eta} \equiv F$ and $-2v_{\xi\eta} \equiv G$ adopt the forms

$$F = \pm \frac{2(a_1 a_2 - a_3 a_0)p_\xi q_\eta}{(a_0 + a_1 p + a_2 q + a_3 pq)^2} \mp \frac{2(b_1 b_2 - b_3 b_0)p_\xi q_\eta}{(b_0 + b_1 p + b_2 q + b_3 pq)^2} \equiv \pm(U_a - U_b) \tag{3.11}$$

$$G = U_a + U_b. \tag{3.12}$$

It is of interest to observe that the localized structures with representations of the type

$$U_a = \frac{2(a_1 a_2 - a_3 a_0)p_\xi q_\eta}{(a_0 + a_1 p + a_2 q + a_3 pq)^2} \tag{3.13}$$

constituent in (3.11), (3.12) have been isolated for a diversity of nonlinear (2 + 1)-dimensional equations [5].

4. Classes of localized excitations of the 2DsG system and their interaction properties

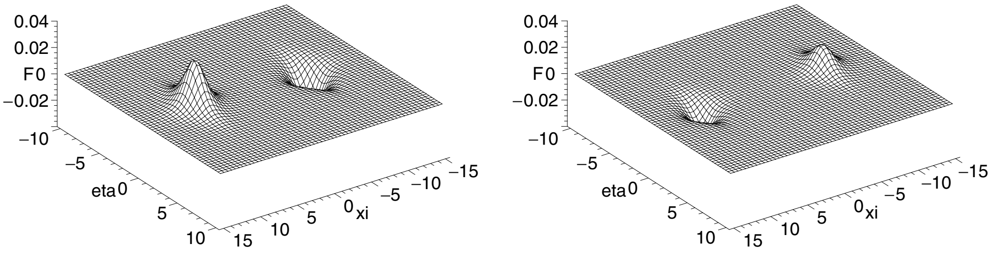
The presence of the arbitrary functions p, q in the representations (3.11), (3.12) implies the existence of a rich diversity of coherent structure solutions of the 2DsG system. Indeed, the spectra of known types of localized solutions including 'inter alia', dromions, breathers, instantons and peakons are all admitted by the 2DsG system by virtue of the representation (3.11)–(3.12).

Clarkson *at al* [7] have shown that a special class of solutions of the 2DsG system may be obtained by a linear superposition of solutions of its (1 + 1)-dimensional counterpart [7]. Thus, if $\{\tilde{u}, \tilde{v}\}$ and $\{\hat{u}, \hat{v}\}$ are two solutions of (1 + 1)-dimensional sG equation, so that

$$\tilde{u}_{zzt} + 2\tilde{u}_z \tilde{v}_{zt} = 0 \qquad \tilde{v}_{zz} = \tilde{u}_z^2 \qquad \hat{u}_{zzt} + 2\hat{u}_z \hat{v}_{zt} = 0 \qquad \hat{v}_{zz} = \hat{u}_z^2 \tag{4.1}$$

then

$$u = \tilde{u}(\xi + \eta, t) + \hat{u}(\xi - \eta, t) \qquad v = \tilde{v}(\xi + \eta, t) + \hat{v}(\xi - \eta, t) \tag{4.2}$$



(*a*) (*b*)

Figure 1. Pre- and post-interaction at $t = -8$ and $t = +8$ of two solitary waves for the potential F of the 2DsG system: p is the (1 + 1)-dimensional kink–antikink solution (4.4) and q is the static single one-dimensional kink solution (4.5) of the (1 + 1)-dimensional sG model.

satisfy the 2DsG system (2.6). By contrast, the representation (3.11), (3.12) determining $u_{\xi\eta}$, $v_{\xi\eta}$ provides a *nonlinear* superposition principle involving arbitrary functions $p(\xi, t)$, $q(\eta, t)$. In particular, the latter may be selected to be solutions of the (1 + 1)-dimensional sine-Gordon equation in spacetimes $\{\xi, t\}$, $\{\eta, t\}$ respectively.

In [5], various types of localized excitations have been discussed for (2 + 1)-dimensional nonlinear systems. However, the interactions of the localized excitations investigated therein do not possess phase shifts. Nevertheless, because of the arbitrariness of the functions p and q, there are, in fact, various types of localized excitations that do suffer phase shifts subsequent to interaction. Possible phase shifts have recently been investigated when the function q is selected as t-independent and p is selected via the relations

$$p = \int^{\xi} p_x x_\xi \, \mathrm{d}\xi \qquad p_x \equiv \sum_{j=1}^{M} f_j(\xi + v_j t) \qquad x = \xi + \sum_{j=1}^{M} g_j(\xi + v_j t) \tag{4.3}$$

where $v_1 < v_2 < \cdots < v_M$ are arbitrary constants and $\{f_j, g_j\}$, $\forall j$ are localized functions with the properties $f_j(\pm\infty) = 0$, $g_i(\pm\infty) = G_i^{\pm} = \text{const}$. In general terms, if the functions p or q are taken as multiple localized solutions that possess the phase shifts of (1 + 1)-dimensional models then the (2 + 1)-dimensional localized solutions involving representations (3.13) inherit phase shift structure.

Localized solutions of the 2DsG system both with and without completely elastic interaction may be generated by taking p, q as multi-soliton or multi-kink solutions of appropriate (1 + 1)-dimensional integrable equations. This is illustrated below in two cases where p, q are taken as particular solutions of (1 + 1)-dimensional sine-Gordon and KdV equations.

Here, the interaction behaviour for the 'potential' F in (3.11) is considered when p is taken as the kink–antikink soliton solution of the usual (1 + 1)-dimensional sG model, which is

$$p = p_{k\bar{k}} = 4 \arctan \frac{\sinh t}{\cosh \xi} \tag{4.4}$$

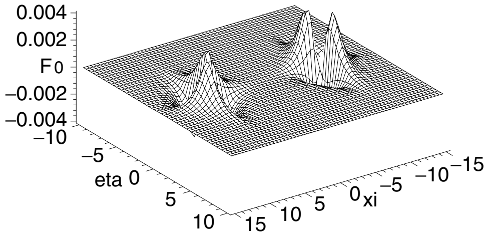
and q is taken as a single static kink solution

$$q = q_k = 4 \arctan \exp \eta. \tag{4.5}$$

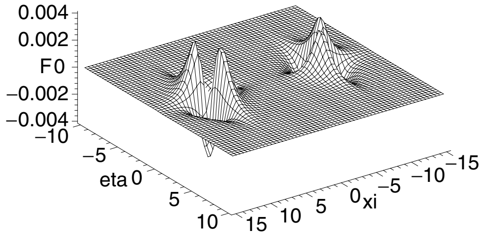
The corresponding parameters in (3.11) read

$$a_0 = b_0 = 20 \qquad a_1 = a_2 = b_1 = b_2 = 1 \qquad a_3 = \tfrac{1}{100} \qquad b_3 = \tfrac{9}{100}. \tag{4.6}$$

Figures 1(*a*) and (*b*) provide plots of the localized excitation of the 2DsG system at times $t = -8$ and $t = +8$ prior to and following interaction respectively. It is evident that amplitudes



(a)



(b)

Figure 2. Pre- and post-interaction of two solitonic excitations (3.11) of the 2DsG system associated with the specification (4.7) of p and q.

are not preserved, so the interaction is not elastic. Although, the direct use of the functions $p_{k\bar{k}}$ and q_k as the functions p and q of (3.11) leads here to non-elastic interaction for the two component localized solutions of the 2DsG system, many kinds of localized solitonic solutions with completely elastic interaction may, in fact, be generated via $p_{k\bar{k}}$ and q_k. Thus, figure 2 illustrates two solitonic excitations with completely elastic interaction behaviour for F with

$$p = (p_{k\bar{k}})_\xi \qquad q = (q_k)_\eta. \tag{4.7}$$

In figure 3, a completely elastic interaction behaviour is exhibited by four localized excitations F of the 2DsG system with parametric values given by (4.6) and both p and q being taken as standard two-soliton solutions of the KdV equation, namely

$$p = -\tfrac{1}{6} + 2\left\{\ln\left[1 + \exp(\xi) + \exp\left(\tfrac{3}{2}\xi - \tfrac{15}{8}t\right) + \tfrac{1}{25}\exp\left(\tfrac{5}{2}\xi - \tfrac{15}{8}t\right)\right]\right\}_{\xi\xi} \tag{4.8}$$

$$q = -\tfrac{1}{6} + 2\left\{\ln\left[1 + \exp(\eta) + \exp\left(\tfrac{3}{2}\eta - \tfrac{15}{8}t\right) + \tfrac{1}{25}\exp\left(\tfrac{5}{2}\eta - \tfrac{15}{8}t\right)\right]\right\}_{\eta\eta}. \tag{4.9}$$

In addition to the completely elastic interaction property, phase shifts are observed following interaction. To reveal the phase shift it has proved convenient to fix one soliton of the KdV equation (both for p and q) possessing zero velocity. Prior to interaction, the smallest soliton is static and situated at $\{\xi = 2\ln 5, \eta = 2\ln 5\}$, the largest soliton is moving with its centre located at $\left\{\xi = \frac{5}{4}t, \eta = \frac{5}{4}t\right\}$ while of the other two coherent solitons one is static and the other is moving. Their centres are located at $\left\{\xi = 2\ln 5, \eta = \frac{5}{4}t\right\}$ and $\left\{\xi = \frac{5}{4}t, \eta = 2\ln 5\right\}$ respectively. From figures 3(e)–(g), it is seen that, following interaction, the static soliton remains static with shape unchanged but its centre is shifted to $\{\xi = 0, \eta = 0\}$. The largest soliton recovers its shape but its centre is shifted to $\left\{\xi = \frac{5}{4}t + 2\ln 5, \eta = \frac{5}{4}t + 2\ln 5\right\}$. As to the other two solitons, they also preserve their shapes and velocities (static in one case) but have their centres shifted to $\left\{\xi = 0, \eta = \frac{5}{4}t + 2\ln 5\right\}$ and $\left\{\xi = \frac{5}{4}t + 2\ln 5, \eta = 0\right\}$, respectively.

In general, if the functions p and q are selected as multi-localized solitonic excitations with

$$p|_{t\to\mp\infty} = \sum_{i=1}^{M} p_i^{\mp} \qquad p_i^{\mp} \equiv p_i\left(\xi - c_i t + \delta_i^{\mp}\right) \tag{4.10}$$

$$q|_{t\to\mp\infty} = \sum_{j=1}^{N} q_j^{\mp} \qquad q_j^{\mp} \equiv q_j\left(\eta - C_j t + \Delta_j^{\mp}\right) \tag{4.11}$$

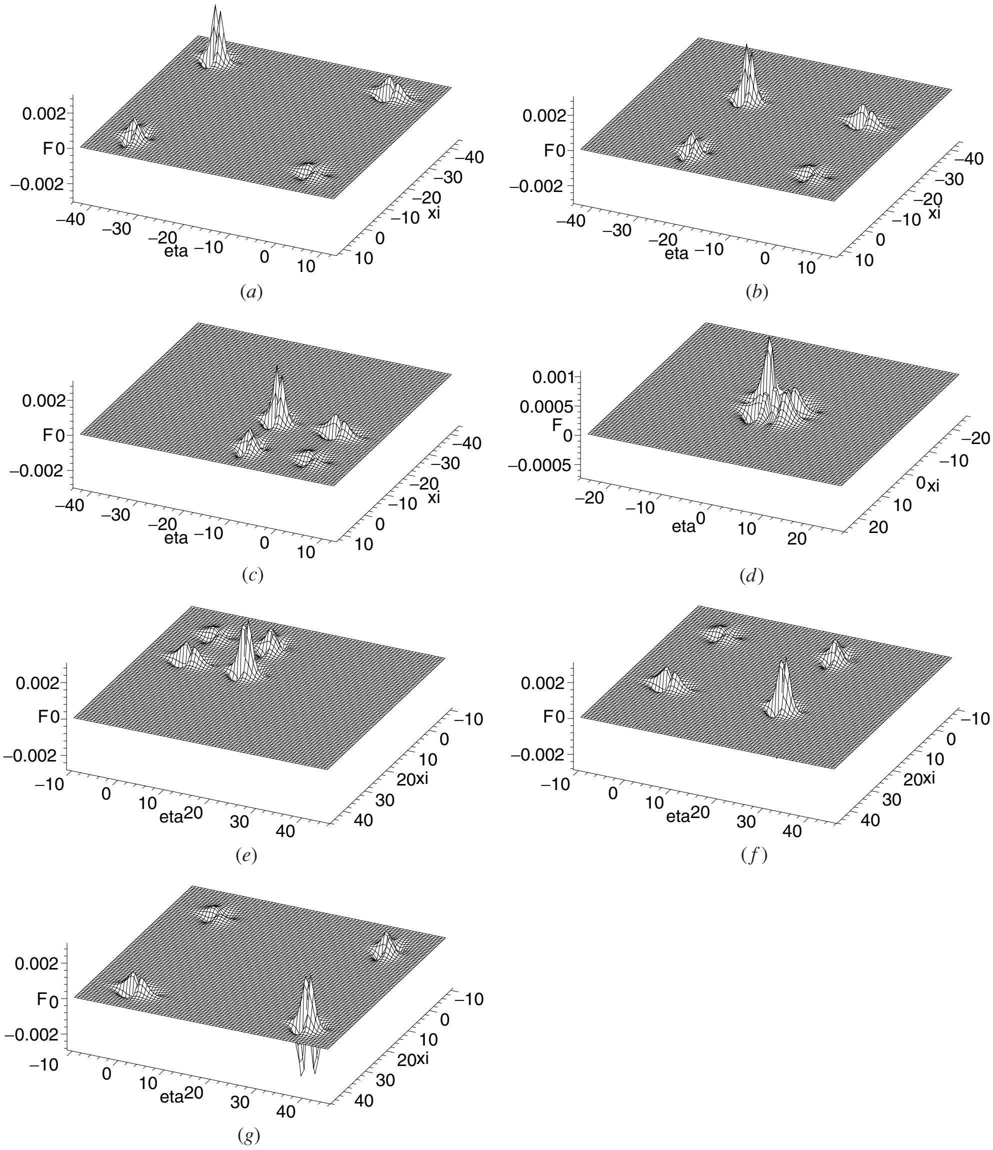


Figure 3. Pre- and post-interaction of a four solitonic solution (3.10) for the 2DsG system; p and q are the two-soliton solutions (4.8) and (4.9) of the KdV equation at times: (a) $t = -28$, (b) $t = -18$, (c) $t = -8$, (d) $t = 0$, (e) $t = 8$, (f) $t = 18$ and (g) $t = 28$.

where $\{p_i, q_j\}\ \forall i$ and j are localized functions, then the potential F expressed by (3.11) delivers $M \times N$ (2 + 1)-dimensional localized excitations with the asymptotic behaviour ($a^2 = 1$)

$$F|_{t\to\mp\infty} \to \sum_{i=1}^{M}\sum_{j=1}^{N}\left\{\frac{2a(a_1a_2 - a_3a_0)\,p_{i\xi}^{\mp}q_{j\eta}^{\mp}}{\left(a_0 + a_1\left(p_i^{\mp} + P_i^{\mp}\right) + a_2\left(q_j^{\mp} + Q_j^{\mp}\right) + a_3\left(p_i^{\mp} + P_i^{\mp}\right)\left(q_j^{\mp} + Q_j^{\mp}\right)\right)^2}\right.$$
$$\left. + \frac{2a(b_1b_2 - b_3b_0)\,p_{i\xi}^{\mp}q_{j\eta}^{\mp}}{\left(b_0 + b_1\left(p_i^{\mp} + P_i^{\mp}\right) + b_2\left(q_j^{\mp} + Q_j^{\mp}\right) + b_3\left(p_i^{\mp} + P_i^{\mp}\right)\left(q_j^{\mp} + Q_j^{\mp}\right)\right)^2}\right\} \qquad (4.12)$$

$$\equiv \sum_{i=1}^{M}\sum_{j=1}^{N} F_{ij}^{\mp}\left(\xi - c_i t + \delta_i^{\mp}, \eta - C_j t + \Delta_j^{\mp}\right) \equiv \sum_{i=1}^{M}\sum_{j=1}^{N} F_{ij}^{\mp} \tag{4.13}$$

where

$$P_i^{\mp} = \sum_{j<i} p_j(\mp\infty) + \sum_{j>i} p_j(\pm\infty) \tag{4.14}$$

$$Q_i^{\mp} = \sum_{j<i} q_j(\mp\infty) + \sum_{j>i} q_j(\pm\infty). \tag{4.15}$$

In the above, it has been assumed, without loss of generality, that $C_i > C_j$ and $c_i > c_j$ if $i > j$.

From the expression (4.12), if $a_i, i = 1, 2, 3$ and $b_i, i = 1, 2, 3$ are fixed as constants, then the *ij*th localized excitation F_{ij} will preserve its shape following interaction iff

$$P_i^+ = P_i^- \qquad Q_j^+ = Q_j^-. \tag{4.16}$$

The phase shifts of the *ij*th localized excitation F_{ij} read

$$\delta_i^+ - \delta_i^- \tag{4.17}$$

in the ξ direction and

$$\Delta_j^+ - \Delta_j^-. \tag{4.18}$$

in the η direction.

From the above discussion, it is seen that multiple localized excitations F, G are readily constructed via (1 + 1)-dimensional multiple localized excitations with the properties (4.10), (4.11) and (4.16). Indeed, any multiple localized solutions (or their derivatives) with completely elastic interaction behaviour of any known (1 + 1)-dimensional integrable models can be used to construct (2 + 1)-dimensional multiple dromion type of solutions for the 2DsG model with completely elastic interaction properties.

In [18], by use of the Lie symmetry approach, it has been established that there exist certain kinds of plateau and basin-type ring localized excitations of the 2DsG system. In fact, multiple travelling plateau and basin-type ring soliton solutions can be generated via the present procedure by taking

$$f = 1 + \mathrm{i}apq \qquad g = 1 - \mathrm{i}apq \tag{4.19}$$

and

$$p = \sum_{i=1}^{M} \alpha_i \exp[r_{1i} - (k_i\xi - c_i t)^2] \qquad q = \sum_{j=1}^{N} \beta_j \exp[r_{2j} - (l_j\eta - C_j t)^2] \tag{4.20}$$

where a, $\alpha_i, \beta_j, k_i, l_j, c_i, C_j, r_{1i}$ and r_{2j} are all arbitrary real constants. The field u is then given by

$$u = \mathrm{i}\ln\frac{f}{g} = 2\arctan(apq) \tag{4.21}$$

$$= 2\arctan\left\{a\sum_{i=1}^{M}\sum_{j=1}^{N}\alpha_i\beta_j\exp[r_{1i} + r_{2j} - (k_i\xi - c_i t)^2 - (l_j\eta - C_j t)^2]\right\}. \tag{4.22}$$

The asymptotic behaviour of the expression (4.22) as $t \to \pm\infty$ indicates elastic interactions of the multiple plateau and basin-type soliton solutions (4.22).

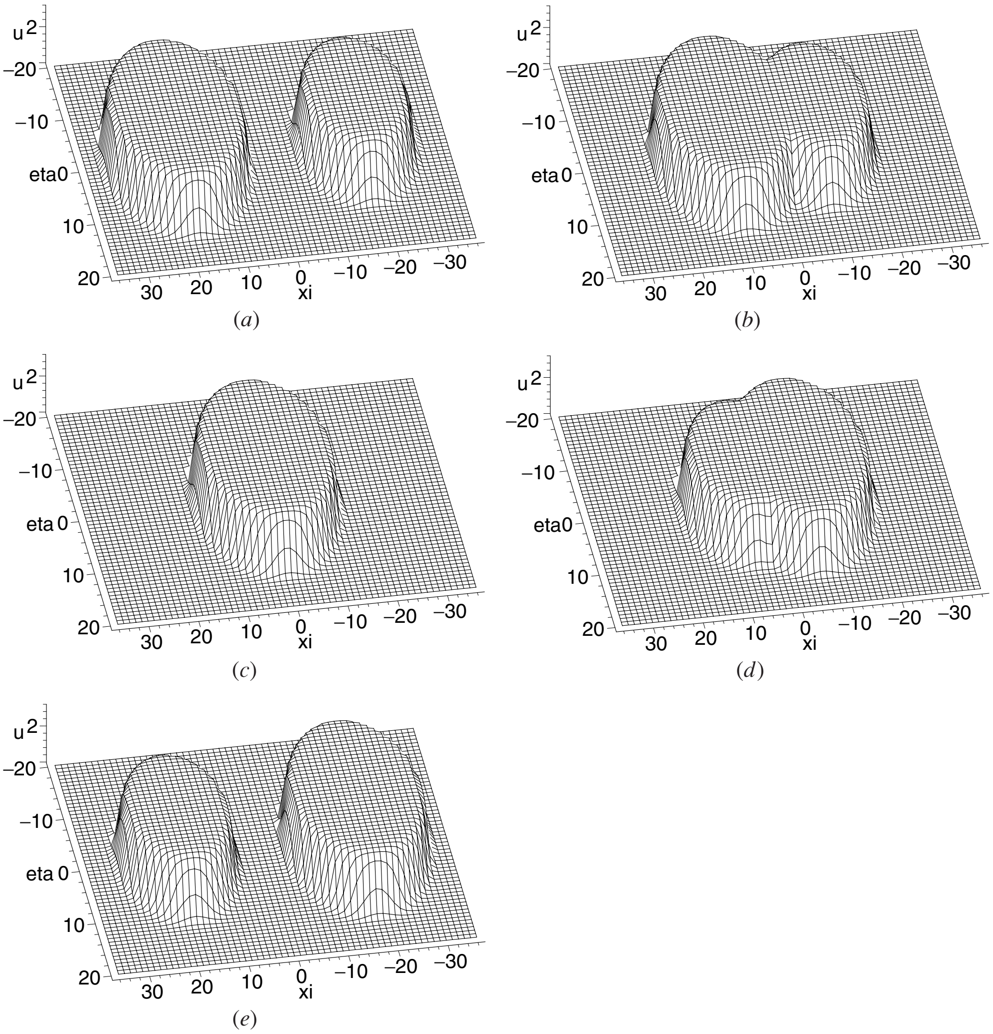


Figure 4. Completely elastic interaction behaviour of two plateau-type ring solitons for the field u determined by (4.22), (4.23) at times: (*a*) $t = -0.9$, (*b*) $t = -0.45$, (*c*) $t = 0$, (*d*) $t = 0.3$ and (*e*) $t = 0.9$.

Figure 4 displays the completely elastic interaction behaviour of two plateau-type solitons expressed by (4.22) with

$$\begin{aligned} &a = 200 \quad M = 2 \quad N = 1 \quad \alpha_1 = 1/3 \quad \alpha_2 = \beta_1 = 1 \quad k_1 = k_2 = l_1 = \frac{1}{\sqrt{10}} \\ &c_2 = -c_1 = \frac{20}{\sqrt{10}} \quad C_1 = 0 \quad r_{11} = 10 \quad r_{12} = 2 \quad r_{21} = 5. \end{aligned} \tag{4.23}$$

From figure 4, it is readily seen that in addition to the completely elastic interaction behaviour, the amplitudes (the heights of the plateaus) of the plateau-type ring solitons are constant in time.

Figure 5 shows the completely elastic interaction behaviour between one plateau-type and one basin-type solitons expressed by (4.22) with

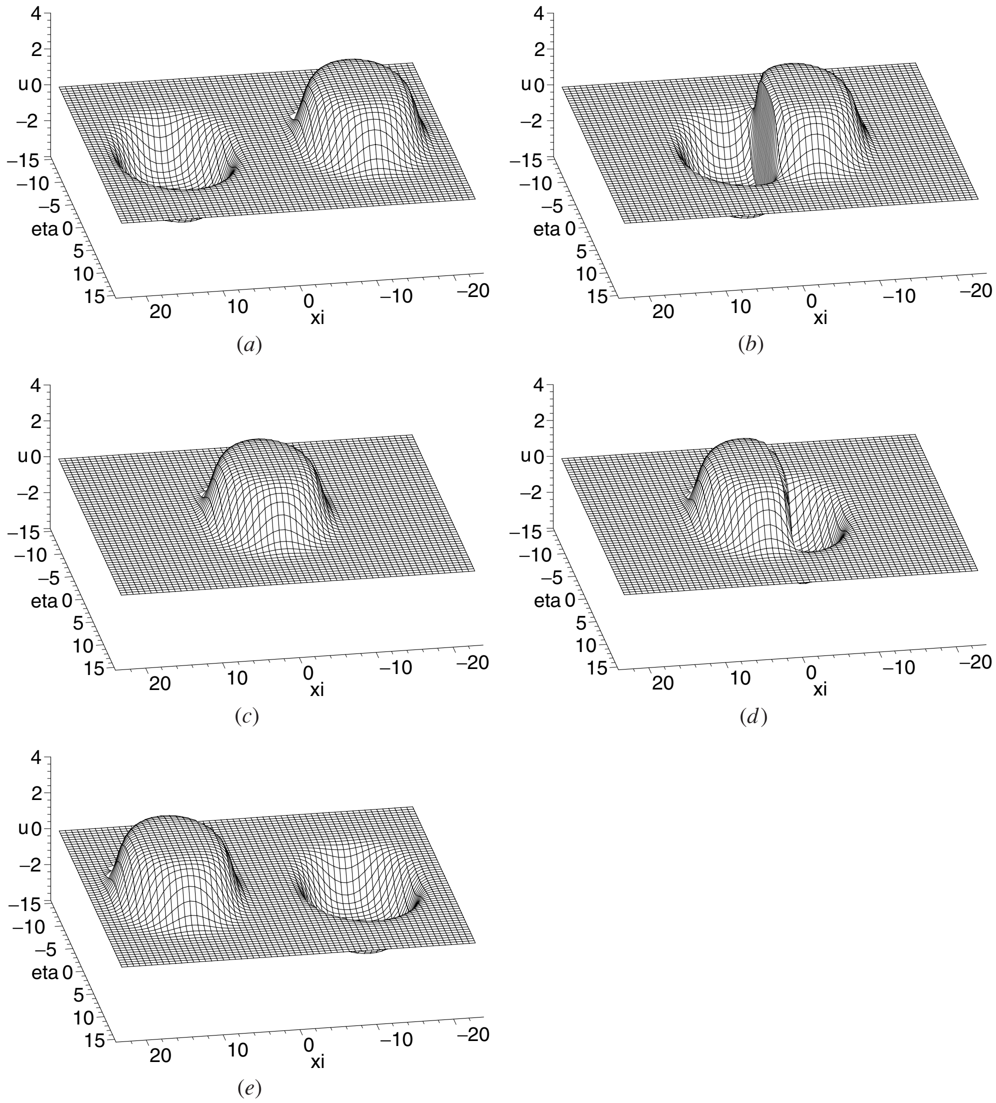


Figure 5. Completely elastic interaction behaviour between plateau type and basin type of ring solitons for the field u determined by (4.22), (4.24) at times: (*a*) $t = -0.6$, (*b*) $t = -0.3$, (*c*) $t = 0$, (*d*) $t = 0.3$ and (*e*) $t = 0.6$.

$$\begin{aligned} &a = 200 \quad M = 2 \quad N = 1 \quad \alpha_1 = -1/3 \quad \alpha_2 = \beta_1 = 1 \quad k_1 = k_2 = l_1 = \tfrac{1}{\sqrt{10}} \\ &c_2 = -c_1 = \tfrac{20}{\sqrt{10}} \quad C_1 = 0 \quad r_{11} = 5 \quad r_{12} = 5 \quad r_{21} = -5. \end{aligned} \tag{4.24}$$

In [18], it has been established that for certain quantities associated with the 2DsG system, bowl-type ring solitons can be constructed. In the present context, it is readily seen that the quantity

$$w \equiv 1 - \cos(2u) \tag{4.25}$$

with u being given by (4.22) represents a multiple bowl-type ring soliton.

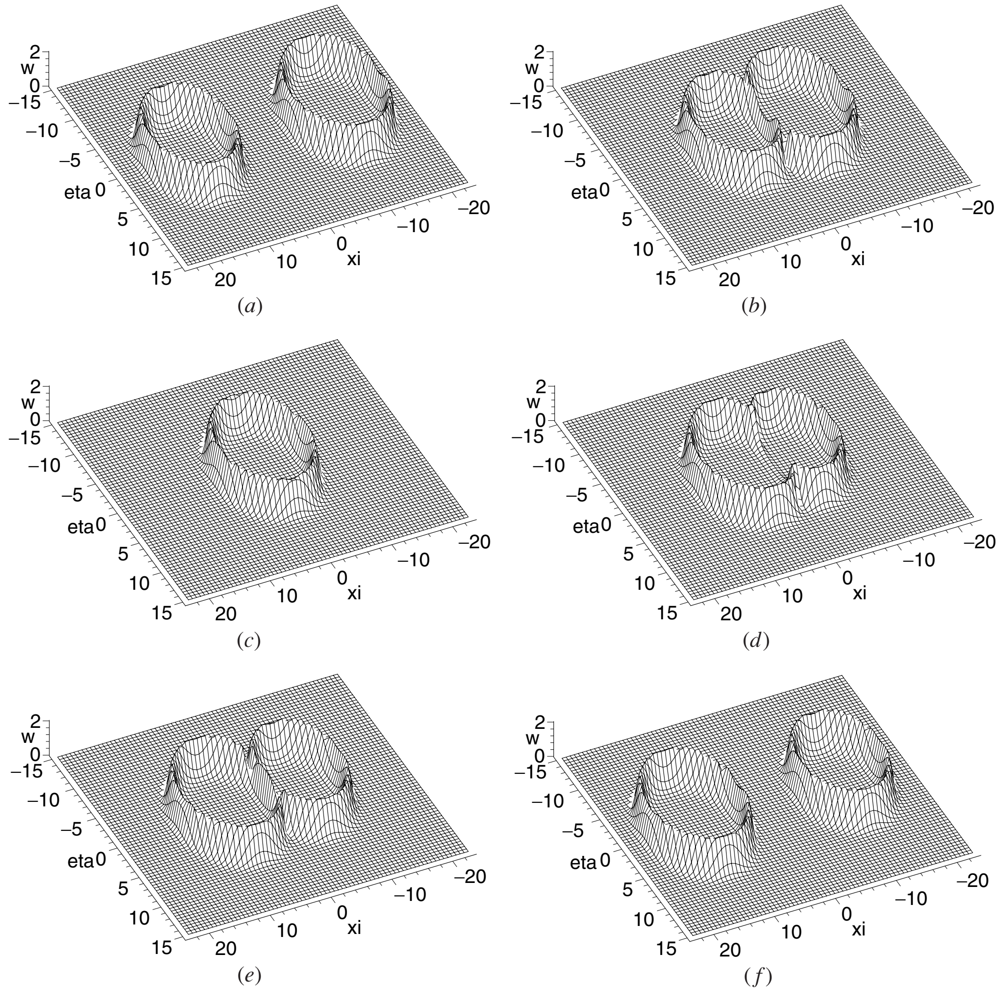


Figure 6. Completely elastic interaction behaviour between two bowl types of ring solitons for the quantity w expressed by (4.25) with (4.22) and (4.24) at times: (*a*) $t = -0.6$, (*b*) $t = -0.27$, (*c*) $t = 0$, (*d*) $t = 0.2$, (*e*) $t = 0.3$ and (*f*) $t = 0.6$.

Figure 6 shows the completely elastic interaction properties of two bowl-type solitons for the quantity w expressed by (4.25) with u given by (4.22) and parametric values as specified in (4.24).

In [5], saddle type of ring solitons was constructed for a range of (2 + 1)-dimensional nonlinear systems. It is of interest to determine whether, in addition to the potentials F and G, there are other quantities that constitute saddle type of ring soliton solutions. Here, in figure 7, the completely elastic interaction behaviour between two particular saddle types of ring solitons for the quantity

$$r \equiv u_\xi u_\eta \tag{4.26}$$

with u being given by (4.22) corresponding to parametric values (4.24) is exhibited.

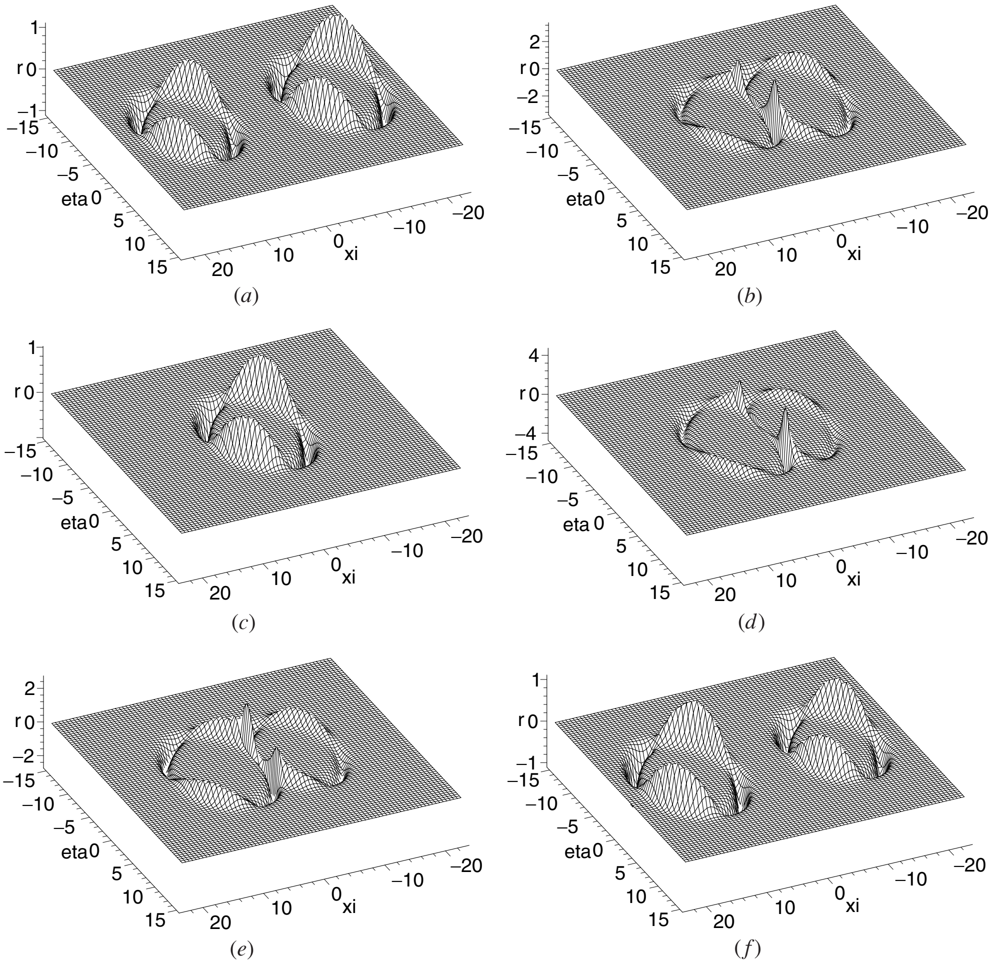


Figure 7. Completely elastic interaction behaviour between two saddle-type ring solitons for the quantity r determined by (4.26) with (4.22) and (4.24) at times: (*a*) $t = -0.6$, (*b*) $t = -0.27$, (*c*) $t = 0$, (*d*) $t = 0.2$, (*e*) $t = 0.3$ and (*f*) $t = 0.6$.

5. Application of the Moutard transformations

In [10], it has been noted that eigenvectors $\Phi = (\varphi_1, \varphi_2)^T$ of (2.7) provide complex solutions $\psi = \varphi_1 + \mathrm{i}\varphi_2$ of the linear equation

$$\psi_{\xi\eta} + U\psi = 0 \tag{5.1}$$

where

$$U = (v - \mathrm{i}u)_{\xi\eta}. \tag{5.2}$$

The classical Moutard transformation (MT) [21] and its modern variants play an important role in soliton theory and its underling geometry [21–23]. The classical MT of (5.1) can be expressed as follows: given solutions ψ and $\theta_1 \neq 0$ of (5.1), then

$$\psi^{(1)} = \frac{S(\theta_1, \psi)}{\theta_1} \tag{5.3}$$

with bilinear potential

$$S(a, b) = \int (ab_x - a_x b)\,\mathrm{d}x + (a_y b - ab_y)\,\mathrm{d}y \tag{5.4}$$

satisfies (5.1) with the potential U replaced by

$$U^{(1)} = U + 2(\ln \theta_1)_{\xi\eta}. \tag{5.5}$$

It is evident that when U transforms according to (5.5), the functions u and v given by (3.1) transform as

$$(\tilde{u}, \tilde{v}) = \left(u + \mathrm{i}\ln\frac{\theta_1}{\bar{\theta}_1}, v + \ln\theta_1\bar{\theta}_1\right) \tag{5.6}$$

where $\{f, g, u_1, v_1\} \to \{\theta_1, \bar{\theta}_1, u, v\}$ and $\bar{\theta}_1$ is linked with θ_1 by (3.2) with the upper sign.

If we again select the seed solution of the 2DsG as (3.4), then the Lax pair (2.7) possesses the general solution

$$\varphi_1 = q(\eta, t) \equiv q \qquad \varphi_2 = -\mathrm{i}p(\xi, t) \equiv -\mathrm{i}p \tag{5.7}$$

$$\psi = \theta_1 = q + p \tag{5.8}$$

with the conditions

$$q_{\eta t} + V_{2\eta t} q = 0 \tag{5.9}$$

$$p_{\xi t} + V_{1\xi t} p = 0. \tag{5.10}$$

Two of $\{p, q, V_1, V_2\}$, say, $\{p, q\}$, can be taken as arbitrary functions.

On substitution of (5.8) into the MT (5.5), and requiring (5.6) to be a solution of the 2DsG system, it is seen that

$$\bar{\theta}_1 = \lambda(t)(p - q) \tag{5.11}$$

where $\lambda(t)$ is an arbitrary function of t. In the following, we select $\lambda(t) = 1$. The next two MTs applied to the linear problem (5.1) yield, in turn,

$$\psi^{(2)} = \frac{S(\theta_1, \theta_2)\psi + S(\theta_2, \psi)\theta_1 + S(\psi, \theta_1)\theta_2}{S(\theta_1, \theta_2)} \tag{5.12}$$

and

$$\psi^{(3)} = \frac{S(\theta_1, \theta_2)S(\theta_3, \psi) + S(\theta_2, \theta_3)S(\theta_1, \psi) + S(\theta_3, \theta_1)S(\theta_2, \psi)}{S(\theta_1, \theta_2)\theta_3 + S(\theta_2, \theta_3)\theta_1 + S(\theta_3, \theta_1)\theta_2} \tag{5.13}$$

with corresponding potentials

$$U^{(2)} = U + 2[\ln S(\theta_1, \theta_2)]_{\xi\eta} \tag{5.14}$$

$$U^{(3)} = U + 2[\ln(S(\theta_1, \theta_2)\theta_3 + S(\theta_2, \theta_3)\theta_1 + S(\theta_3, \theta_1)\theta_2)]_{\xi\eta}. \tag{5.15}$$

For the related solutions of the 2DsG system with the seed solution U given by (3.4), we have

$$\theta_i = p_i(\xi, t) + q_i(\eta, t) \equiv p_i + q_i \tag{5.16}$$

$$\bar{\theta}_i = p_i(\xi, t) - q_i(\eta, t) \qquad i = 1, 2, 3 \tag{5.17}$$

$$q_{i\eta t} + V_{2\eta t} q_i = 0 \qquad p_{i\xi t} + V_{1\xi t} p_i = 0 \tag{5.18}$$

$$S(\theta_i, \theta_j) = 2\int p_i p_{j\xi}\,\mathrm{d}\xi - 2\int q_i q_{j\eta}\,\mathrm{d}\eta - (p_i - q_i)(p_j + q_j) \tag{5.19}$$

$$\bar{S}(\bar{\theta}_i, \bar{\theta}_j) = 2\int p_i p_{j\xi}\,\mathrm{d}\xi - 2\int q_i q_{j\eta}\,\mathrm{d}\eta - (p_i + q_i)(p_j - q_j). \tag{5.20}$$

$$u^{(2)} = \mathrm{i}\ln\frac{S(\theta_1, \theta_2)}{\bar{S}(\bar{\theta}_1, \bar{\theta}_2)} \tag{5.21}$$

$$v^{(2)} = V_1 + V_2 + \ln[S(\theta_1, \theta_2)\bar{S}(\bar{\theta}_1, \bar{\theta}_2)] \tag{5.22}$$

$$\begin{aligned} u^{(3)} = {} & \mathrm{i}\ln[S(\theta_1, \theta_2)\theta_3 + S(\theta_2, \theta_3)\theta_1 + S(\theta_3, \theta_1)\theta_2] \\ & - \mathrm{i}\ln[\bar{S}(\bar{\theta}_1, \bar{\theta}_2)\bar{\theta}_3 + \bar{S}(\bar{\theta}_2, \bar{\theta}_3)\bar{\theta}_1 + \bar{S}(\bar{\theta}_3, \bar{\theta}_1)\bar{\theta}_2] \end{aligned} \tag{5.23}$$

$$\begin{aligned} v^{(3)} = {} & \ln[S(\theta_1, \theta_2)\theta_3 + S(\theta_2, \theta_3)\theta_1 + S(\theta_3, \theta_1)\theta_2] \\ & + \ln[\bar{S}(\bar{\theta}_1, \bar{\theta}_2)\bar{\theta}_3 + \bar{S}(\bar{\theta}_2, \bar{\theta}_3)\bar{\theta}_1 + \bar{S}(\bar{\theta}_3, \bar{\theta}_1)\bar{\theta}_2] + V_1 + V_2. \end{aligned} \tag{5.24}$$

It is emphasized that, for the complex 2DsG system, p_i and q_i may be complex functions, then $\bar{\theta}_i$ and $\bar{S}(\bar{\theta}_i, \bar{\theta}_j)$ are not generally proportional to the complex conjugates of θ_i and $S(\theta_i, \theta_j)$. However, for the real 2DsG system, $\bar{\theta}_i$ and $\bar{S}(\bar{\theta}_i, \bar{\theta}_j)$ must be proportional to the complex conjugates of θ_i and $S(\theta_i, \theta_j)$. And for the real 2DsG system the results of the Moutard transformation have been given by many authors [8, 10, 13, 14].

The final fact that should be pointed out is that the variable separation solution (3.1) with (3.4), (3.6)–(3.8) obtained by the multi-linear variable separation approach is equivalent to that of a special case of the second step Moutard transformation (the author is indebted to Wolfgang Schief for this observation) by taking $p_1 = q_2 = 0$,

$$q_1 = \frac{\alpha_1 q + \alpha_0}{\alpha_2 q + \alpha_3} \qquad p_2 = \frac{\beta_1 p + \beta_0}{\beta_2 p + \beta_3}$$

with the suitable selections of the constants $\alpha_i, \beta_i, i = 1, 2, 3, 4$ and the redefinitions of the functions V_1 and V_2.

6. Summary and discussion

In summary, some types of variable separation solutions can be obtained for the 2DsG system via some different ways such as the multi-linear variable separation approach and the Moutard transformation. The usual 'universal' quantity valid for many other (2 + 1)-dimensional systems is included as a special example of the variable separation solutions for the potentials F and G. For the 2DsG model, two (2 + 1)-dimensional exact solutions with some special conditions can be linearly combined to get new exact solutions. In the variable separation solutions obtained from the multi-linear variable separation approach, there are two (1 + 1)-dimensional arbitrary functions. The more variable separated functions can be included in the variable separation solutions via Moutard transformation. The Moutard transformations are given generally for the complex 2DsG system. Whence the real condition of the 2DsG system is used, the Moutard transformations given here reduce back to the known ones [13]. The variable separation solutions obtained by the multi-linear variable separation approach can be considered as an equivalent special case of the second step Moutard transformation.

Though many kinds of localized excitations for a diversity of (2 + 1)-dimensional models have been obtained from the variable separation solutions [5], it is still not clear to construct the solitonic solutions such that the interactions are completely elastic. In this paper, a general convenient method is proposed to construct infinitely many kinds of multiple localized excitation with completely elastic interaction behaviours and this method is valid for all the models listed in [5]. Especially, for the 2DsG system, it is proved that the plateau type, basin type, bowl type and saddle type of ring shape excitations [18] are really solitons because of their completely interacting behaviours.

There are various important problems for (2 + 1)-dimensional integrable systems which are still open and worth studying further. For instance, for the 2DsG system two universal terms (every term is related to an exact solution) can be linearly combined to construct new exact solutions. Are there any other systems possessing this property? Though the variable separation approach has been applied to various (2 + 1)-dimensional integrable systems, some of other (2 + 1)-dimensional integrable systems, especially, the Kadomtsev–Petviashvili equation and the Sawada–Kortera model have not yet been solved via this approach. Can we solve all the (2 + 1)-dimensional integrable models via multiple linear variable separation approach? Are there any relations among the variable separation approach solvability and the usual integrability?

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